Exact Computation of the Entanglement Entropy of Free Fermions on Hamming graphs

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Entanglement entropy

Its role: Measuring how much two regions of a system are entangled (or intertwined).

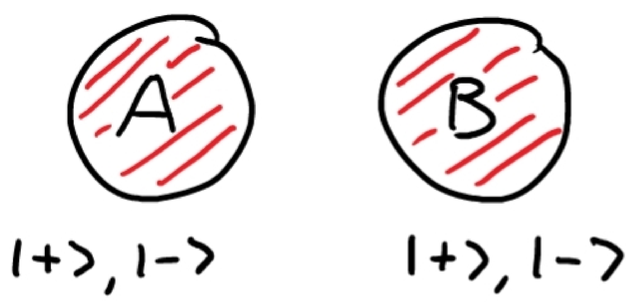
$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$$

Formula:

$$\rho_A = \mathrm{tr}_B \left|\psi\right\rangle \left\langle\psi\right| \quad \text{(reduced density matrix)}$$

$$S = -\mathrm{tr}\rho_A \ln \rho_A \quad \text{(Von Neumann entropy)}$$

Example:



$$\left|\psi\right\rangle = \frac{1}{\sqrt{2}}\left|+\right\rangle_A \otimes \left|-\right\rangle_B + \frac{1}{\sqrt{2}}\left|-\right\rangle_A \otimes \left|+\right\rangle_B \rightarrow S \approx 0.69$$

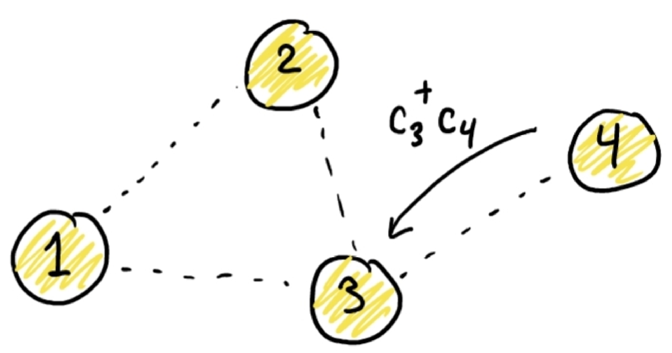
The problem: Considering large system (thermodynamic limit) requires a lot of computing power.
Our goal: Obtaining analytical expressions for specific systems

Free fermions

Hamiltonian:

$$\hat{H} = \sum_{k,k'} \alpha_{k,k'} c_k^\dagger c_k = c^\dagger A\, c \qquad \begin{aligned} \{c_k, c_{k'}\} &= 0, \\ \{c_k^\dagger, c_{k'}^\dagger\} &= 0, \\ \{c_k, c_{k'}^\dagger\} &= \delta_{kk'}. \end{aligned}$$

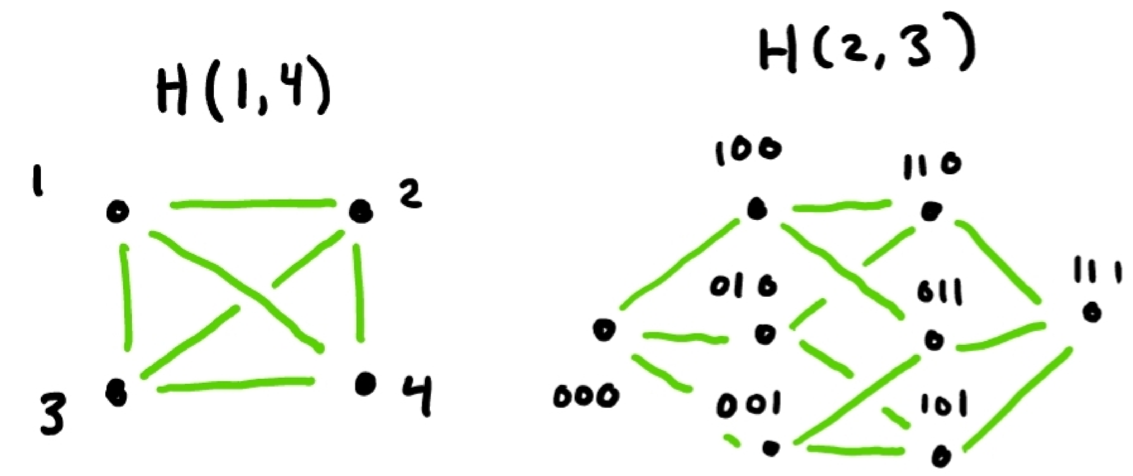
where A is a symmetric matrix and the $c_k^\dagger$ and c_k are fermionic creation and annihilation operators. $c^\dagger$ and c are vectors of operators.



$$\underbrace{\widehat{H} = \sum_{k=0}^{d} \sum_{l=1}^{D_k} \Omega_k \bar{c}_{kl}^\dagger \bar{c}_{kl}}_{\text{diagonalized}}.$$

Hamming graphs $H(d,q)$

Vertices: we take the set of tuples v of length d made of elements in the set $\{0, 1, \dots, q-1\}$. **Edges:** the tuples $v = (v_1, \dots, v_d)$ and $v' = (v_1', \dots, v_d')$ are connected by an edge if there exists a unique i such that $v_i \neq v_i'$.



The subsystems: we want the entropy for the sets SV of vertices at a distance in SD from a given vertex. **The state:** we take the system to be in its ground state $|\Psi_0\rangle\rangle$ which is obtained by filling up the fermi sea.

Chopped correlation matrix

It is defined as

$$C_{vv'} = \langle\langle \Psi_0 | c_v^\dagger c_{v'} | \Psi_0 \rangle\rangle, \qquad \text{where } v, v' \in SV.$$

$|\Psi_0\rangle\rangle$ is a slater determinant $\rightarrow$ the density matrix ρ_A is expressible as the exponential of an entanglement hamiltonian h which commute with C [1]. Thus, if λ are eigenvalues of C

$$S = -\sum_{\lambda} D_\lambda \left[\lambda \ln(\lambda) + (1-\lambda)\ln(1-\lambda)\right].$$

In terms of projection operators E_i over the eigenspace of A and the projection operators E_i^* over sites at a fixed distance, we have

$$C = \sum_{i,i' \in SD} \sum_{k \in SE} E_i^* E_k E_{i'}^*$$

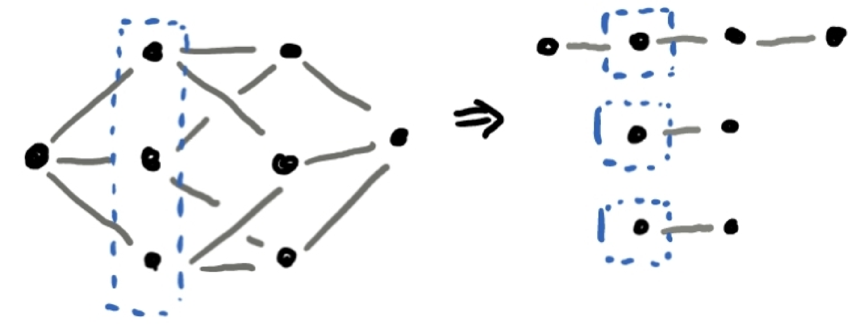
This can be studied from an algebraic approach. Indeed, the Hamming graphs are **distance regular**.

$$AA_i = iA_{i-1} + (d-i)(q-1)A_{i+1}$$

Thus, they define $P-$ and $Q-$ polynomials association schemes. In particular, their projection operators are associated to an algebra referred to as its **Terwilliger algebra** $\mathcal{T}$.

Terwilliger algebra

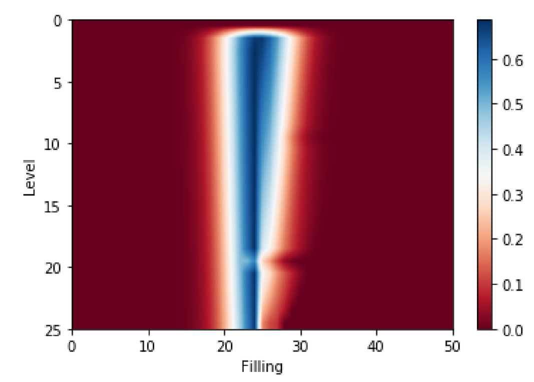
The decomposition in $\mathcal{T}-$ irreducible modules amounts decomposition in chains.



This is done by identifying $\mathcal{T}$ as $\mathfrak{su}(2)$ + central charge. The dimension of C for a chain is smaller. For SV = one level:

$$\lambda_{n,j} = \sum_{k \in SE} Q^2_{d-\frac{n}{2}-i,\ j-k'+\frac{n}{2}},$$

$$Q_{m,k'} \sim K_{j-m}(k'; \frac{q-1}{q}, 2j).$$



Heun operator and the algebraic Bethe ansatz

The approach developped to study **time and band limiting** problems can be applied to identified a block-tridiagonal operator T such that $[C, T] = 0$ [2].

$$T \sim \alpha\{s^x, s^z\} + \beta\{s^z, s^z\} + \gamma s^x + \omega s^z + \zeta$$

This is identified as a $BC-$ Gaudin magnet Hamiltonian in a magnetic field. It can be diagonalized by the **algebraic Bethe ansatz**.

Final remarks

- Considering free fermions on distance regular graph allow one to use the tools from **algebraic combinatorics** and the study **time and band limiting** to compute entropies.
- This approach can be used for any distance regular graph. Next : **the Johnson graphs**.

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References: [1] I. Peschel. Journal of Physics A: Mathematical and General, 36(14):L205–L208,Mar 2003. [2] N. Crampé, K. Guo, and L. Vinet. Nuclear Physics B, 960:115176, 2020.